MA 3113 Practice Exam 1 Solutions

1.) Consider the dependence eq: $c_1\begin{pmatrix}1\\2\\3\end{pmatrix}+c_2\begin{pmatrix}4\\5\\6\end{pmatrix}+c_3\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}1&4&2&0\\2&5&1&0\\3&6&0&0\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c}1&0&-2&0\\0&1&1&0\\0&0&0&0\end{array}\right)$

$\Rightarrow \begin{array}{l}c_1 = 2c_3\\ c_2=-c_3\\ c_3=c_3\end{array}$ so c_3 free variable $\therefore$ lin. dep.

2.) Consider the dependence eq: $c_1\begin{pmatrix}5\\0\\0\end{pmatrix}+c_2\begin{pmatrix}7\\2\\-6\end{pmatrix}+c_3\begin{pmatrix}9\\4\\-8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c}5&7&9&0\\0&2&4&0\\0&-6&-8&0\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right)$

$\Rightarrow c_1=c_2=c_3=0$ $\therefore$ lin. indep.

3.) Let $f, g \in C'(\mathbb{R})$. Then $T(f+g)=\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}=T(f)+T(g)$. Let $c\in\mathbb{R}$. Then $T(cf)=\frac{d}{dx}(cf)=c\frac{df}{dx}=cT(f)$

$\therefore T$ is linear.

4.) Let $f, g\in C(\mathbb{R})$. Then $T(f+g)=\sum_{j=1}^{n}(f+g)(x_j)=\sum_{j=1}^{n}\left(f(x_j)+g(x_j)\right)=\sum_{j=1}^{n}f(x_j)+\sum_{j=1}^{n}g(x_j)=T(f)+T(g)$. Let $c\in\mathbb{R}$

then $T(cf)=\sum_{j=1}^{n}(cf)(x_j)=\sum_{j=1}^{n}cf(x_j)=c\sum_{j=1}^{n}f(x_j)=cT(f)$. $\therefore T$ is linear.

5.) Let $x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$ and $e_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $e_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$, $e_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $e_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, then $x=x_1e_1+x_2e_2+x_3e_4$. Since T is lin, compute $T(e_1)$, $T(e_2)$

$T(e_3)$ and $T(e_4)$. Then $T(e_1)=\begin{pmatrix}2\\0\\0\\0\end{pmatrix}$, $T(e_2)=\begin{pmatrix}-1\\10\\4\\11\end{pmatrix}$, $T(e_3)=\begin{pmatrix}0\\2\\0\\0\end{pmatrix}$, $T(e_4)=\begin{pmatrix}0\\0\\5\\-8\end{pmatrix}$. $\therefore A=\begin{pmatrix}2&-1&0&0\\0&10&2&0\\0&4&0&5\\0&11&0&-8\end{pmatrix}$

6.) Let $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, $e_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$, then $x=x_1e_1+x_2e_2+x_3e_3$. Since T is lin., compute $T(e_1)$, $T(e_2)$, $T(e_3)$.

So $T(e_1)=\begin{pmatrix}1\\0\end{pmatrix}$, $T(e_2)=\begin{pmatrix}4\\3\end{pmatrix}$, $T(e_3)=\begin{pmatrix}5\\-2\end{pmatrix}$. $\therefore A=\begin{pmatrix}1&4&5\\0&3&-2\end{pmatrix}$

7.) Let $B=\begin{pmatrix}b_1&b_2\\b_3&b_4\end{pmatrix}$, then $AB=\begin{pmatrix}3b_1-6b_3 & 3b_2-6b_4\\-b_1+2b_3 & -b_2+2b_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow \begin{cases}3b_1-6b_3=0\\-b_1+2b_3=0\end{cases}$, $\begin{cases}3b_2-6b_4=0\\-b_2+2b_4=0\end{cases}$

pick $b_3=1 \Rightarrow b_1=2$, pick $b_4=2\Rightarrow b_2=4 \Rightarrow B=\begin{pmatrix}2&4\\1&2\end{pmatrix}$

8.) Consider $AB=\begin{pmatrix}23&-10+5k\\-9&15+k\end{pmatrix}$ and $BA=\begin{pmatrix}23&15\\6-3k&15+k\end{pmatrix}$. Want $AB=BA$. Comparing entries get $\begin{cases}6-3k=-9\\-10+5k=15\end{cases}$

Solving for k $\Rightarrow k=5$.

9.) Consider $(B-C)D=0 \Rightarrow BD-CD=0 \Rightarrow BD=CD$. Since D^{-1} exists $\Rightarrow (BD)D^{-1}=(CD)D^{-1}\Rightarrow B(DD^{-1})=C(DD^{-1})$

$\Rightarrow B\cdot I=C\cdot I \Rightarrow B=C$

10.) Suppose $\lambda=0$. then $(A-\lambda I_n)v=0$ has nonzero soln. Since $\lambda=0 \Rightarrow Av=0$ has nontrivial soln, ie $v\neq 0$

$\Rightarrow A$ has nontrivial nullspace. But A is invertible meaning only soln to $Av=0$ is $v=0$. clear contradiction

$\therefore \lambda\neq 0$.

11.) setup $[A \mid I_3] = \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 4R_2} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right) \xrightarrow{R_3 + 3R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right)$

$\xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right) \xrightarrow[R_2 - 3R_3]{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & -2 & 4 & -1 \\ 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right) \xrightarrow{\text{swap rows}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right)$

$\therefore \; A^{-1} = \begin{pmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{pmatrix}$

12.) setup $[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 + 3R_1]{R_3 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 3 & 1 \end{array}\right)$

$\xrightarrow[R_2 + R_3]{R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & 1 \\ 0 & 1 & 0 & 7 & 4 & 1 \\ 0 & 0 & 2 & 4 & 3 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & 1 \\ 0 & 1 & 0 & 7 & 4 & 1 \\ 0 & 0 & 1 & 2 & \frac{3}{2} & \frac{1}{2} \end{array}\right) \quad \therefore \; A^{-1} = \begin{pmatrix} 5 & 3 & 1 \\ 7 & 4 & 1 \\ 2 & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

13.) It is not always true. Consider $A = 0$. Then A is $n \times n$ and $Av_j = 0$ for $j = 1, \dots, k$

$\therefore \; Av_1, \dots, Av_k$ are not lin. indep.